B.Tech 3rd Semester Exam., 2020
(New Course)

MATHEMATICS—III

(Differential Calculus)

Time : 3 hours *Full Marks* : 70

Instructions :

(i) *The marks are indicated in the right-hand margin.*

(ii) *There are **NINE** questions in this paper.*

(iii) *Attempt **FIVE** questions in all.*

(iv) *Question No. **1** is compulsory.*

1. Choose the correct answer of the following (any *seven*) : 2×7=14

(a) The value of $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x}$ is

(i) 0

(ii) 1

(iii) e

(iv) $1/e$

(b) Let $f(x) = |x|$ and $g(x) = |x^3|$, then

(i) $f(x)$ and $g(x)$ both are continuous at $x = 0$

(ii) $f(x)$ and $g(x)$ both are differentiable at $x = 0$

(iii) $f(x)$ is differentiable but $g(x)$ is not differentiable at $x = 0$

(iv) $f(x)$ and $g(x)$ both are not differentiable at $x = 0$

(c) The value of $\nabla^2[(1-x)(1-2x)]$ is equal to

(i) 2

(ii) 3

(iii) 4

(iv) 6

(d) If $v = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$, then the value of curl v at (1, −1, 1) is equal to

(i) $-(\hat{j} - 2\hat{k})$

(ii) $(\hat{i} + 3\hat{k})$

(iii) $-(\hat{i} + 2\hat{k})$

(iv) $(\hat{i} + 2\hat{j} + \hat{k})$

(e) The degree of the differential equation

$$y\frac{dx}{dy}+\left(\frac{dx}{dy}\right)^2+\sin y\left(\frac{dx}{dy}\right)^3-\cos x=0$$

is

(i) 0

(ii) 1

(iii) 2

(iv) Cannot be determined

(f) The solution of the boundary value problem

$$(x-y^2x)\,dx+(x^2y-y)\,dy=0,\ y(0)=0$$

is

(i) $x^2-y^2=0$

(ii) $2x-y=0$

(iii) $x-2y=0$

(iv) None of the above

(g) Let $P_n(x)$ be the Legendre polynomial of degree $n\geq 0$. If

$$\int_{-1}^{1}P_{n-1}^2(x)\,dx=\frac{2}{(kn-l)}$$

then the value of (k, l) is

(i) (1, 1) (ii) (1, 2)

(iii) (2, 1) (iv) (2, 2)

(h) The general solution of Bessel differential equation

$$x^2y''(x)+xy'(x)+(x^2-64)y(x)=0$$

is

(i) $y=AJ_8(x)+BJ_{-8}(x)$, where A and B are arbitrary constants

(ii) $y=AJ_8(x)+BY_{-8}(x)$, where A and B are arbitrary constants

(iii) $y=AJ_8(x)+J_{-8}(x)$, where A is arbitrary constant

(iv) $y=J_{3/4}(x)+Y_{3/4}(x)$

(i) The equation $p\tan y+q\tan x=\sec^2 z$ is of order

(i) 1

(ii) 2

(iii) 0

(iv) None of the above

(j) The solution of $p\tan x+q\tan y=\tan z$ is

(i) $\sin x/\sin y=\varphi(\sin y/\sin z)$

(ii) $\sin x.\sin y=\varphi(\sin y/\sin z)$

(iii) $\sin x/\sin y=\varphi(\sin y,\ \sin z)$

(iv) $\sin x/\sin y=\varphi(\sin y.\sin z)$

2. (a) If $y = (\sin^{-1} x)^2$, then show that

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$$

Hence find $(y_n)_0$. 7

(b) Find the value of

$$\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}$$ 7

3. (a) Discuss the continuity of the following function $f(x, y)$ at point $(0, 0)$: 7

$$f(x, y) = \begin{cases} \dfrac{\sin\sqrt{|xy|} - \sqrt{|xy|}}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$

(b) For the function

$$f(x, y) = \begin{cases} \dfrac{xy(2x^2 3y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$

check whether $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ are equal or not. 7

4. (a) Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$. 8

(b) Obtain the second-order Taylor's series approximation to the function

$$f(x, y) = xy^2 + y\cos(x - y)$$

about the point (1, 1). 6

5. (a) If $f = (x^2 + y^2 + z^2)^{-n}$, then find div grad f and determine n, if div grad $f = 0$. 6

(b) Verify Green's theorem for

$$\int_C [(xy + y^2)dx + x^2 dy]$$

where C is bounded by $y = x$, $y = x^2$. 8

6. (a) Find the value of n for which the vector $r^n \mathbf{r}$ is solenoidal, where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$. 7

(b) Solve the differential equation

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$$ 7

7. Solve the following differential equations : 7+7=14

(a) $p = \sin(y - xp)$. Also find its singular solution.

(b) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x \log x$

8. *(a)* Prove that

$$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x))$$ 6

(b) Prove that

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} P_n(1) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$ 8

9. Solve the following differential equations : 7+7=14

(a) $x^2p + y^2q = (x+y)z$

(b) $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$

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